Forum for Electromagnetic Research Methods and Application Technologies (FERMAT)

Antenna current optimization and physical bounds for small antennas

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Abstract: Antenna current optimization is used to determine optimal currents in the antenna design region. The currents provide understanding, physical bounds, and figures of merits for antenna designs. The methodology is particularly useful for antennas that are constrained by their electrical size such as small antennas. The antenna design optimization problem is formulated as a convex optimization problem expressed in the currents on the antenna structure. This presentation reviews antenna current optimization and numerical results for maximization of the gain to Q-factor quotient and minimization of the Q-factor are presented.

Keywords: Small antennas, physical bounds, Q-factor, stored energy, convex optimization, antenna current optimization

Mats Gustafsson received the M.Sc. degree in Engineering Physics 1994, the Ph.D. degree in Electromagnetic Theory 2000, was appointed Docent 2005, and Professor of Electromagnetic Theory 2011, all from Lund University, Sweden. He co-founded the company Phase holographic imaging AB in 2004. His research interests are in scattering and antenna theory and inverse scattering and imaging with applications in microwave tomography and digital holography. He has written over 80 peer reviewed journal papers and 100 conference papers. Prof. Gustafsson received the Best Antenna Poster Prize at EuCAP 2007, the IEEE Schelkunoff Transactions Prize Paper Award 2010, and the Best Antenna Theory Paper Award at EuCAP 2013. He served as an IEEE AP-S Distinguished Lecturer for 2013-15.

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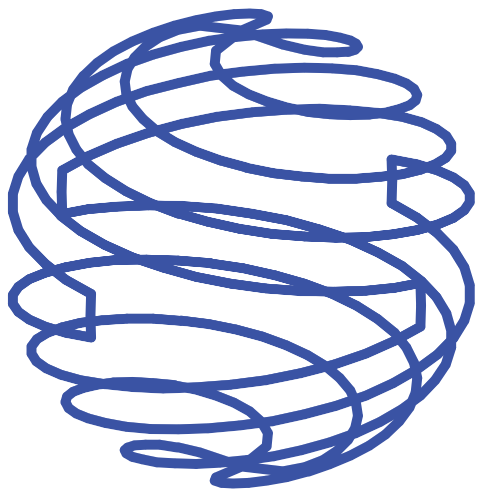
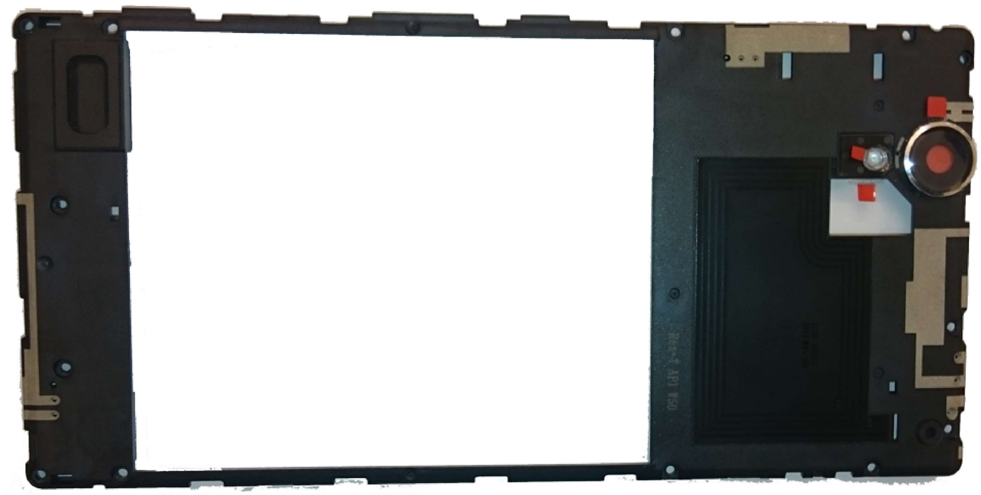
Small antennas



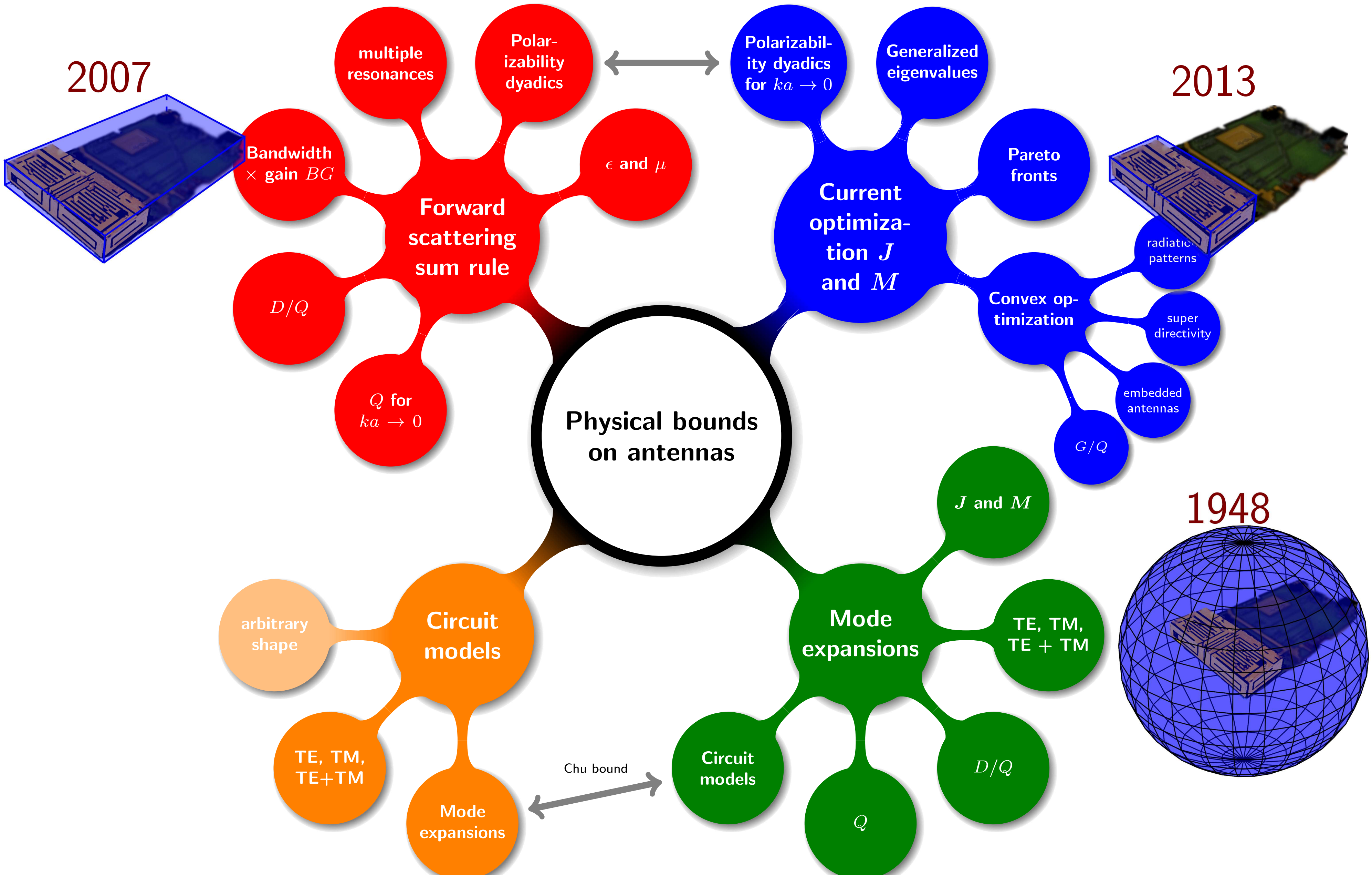
Folded spherical helix



Sony Xperia

- Many advanced small antenna designs.
- Antennas embedded in structures.
- Performance in Q, bandwidth and efficiency.
- How does the performance depend on the design region?
- Understanding from bounds and optimal currents.
- Automated optimal antenna design?

Physical bounds on antennas: methods



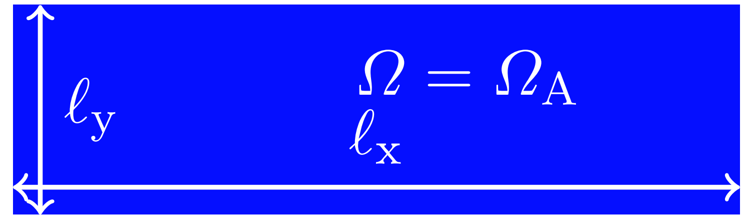
Gustafsson *etal*, Physical Bounds of Antennas in Handbook of Antenna Technologies, 2016

Antenna and antenna current optimization

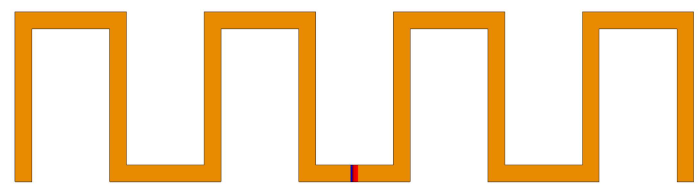
Device structure Ω with a maximal size for the antenna region Ω_{A}.

- **Antenna optimization:** determine the shape, material, and feed properties for optimal performance.
- **Antenna current optimization:** synthesize an optimal current distribution in the available geometry.

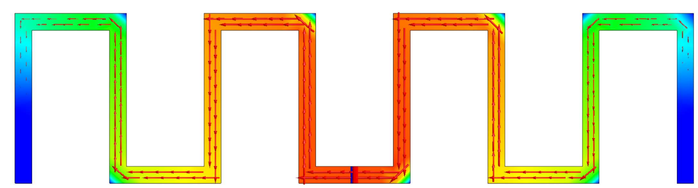
Maximal size of the antenna



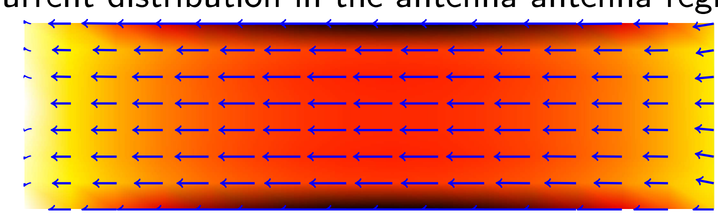
Antenna geometry with feed point



Current distribution on the antenna



Current distribution in the antenna antenna region



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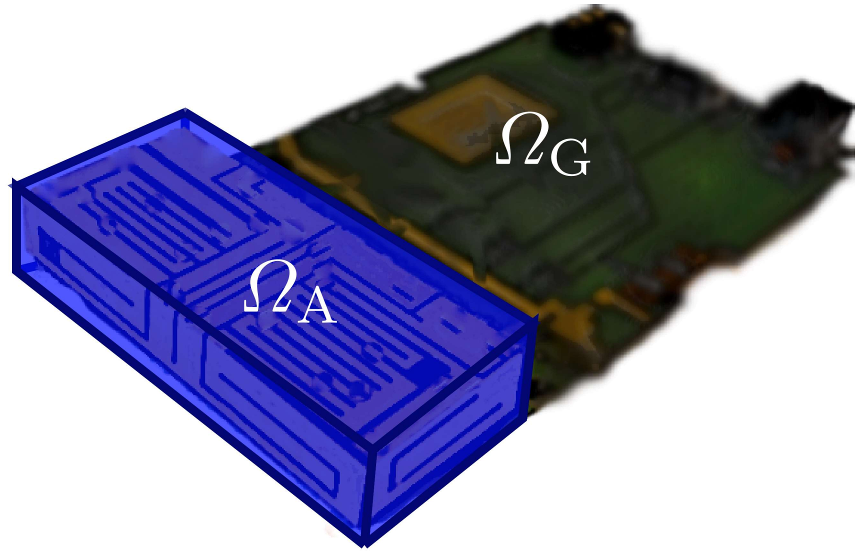
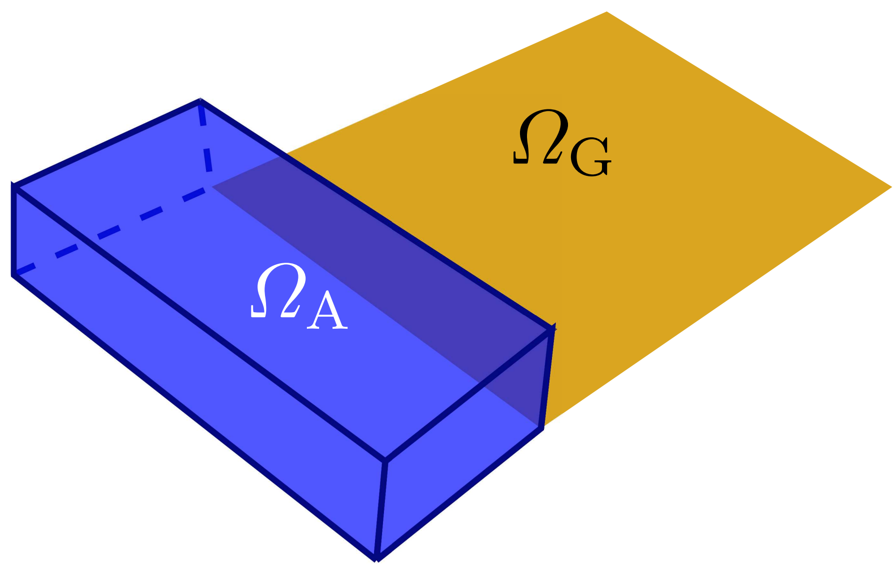




Optimization of antenna currents: examples

Gain over Q

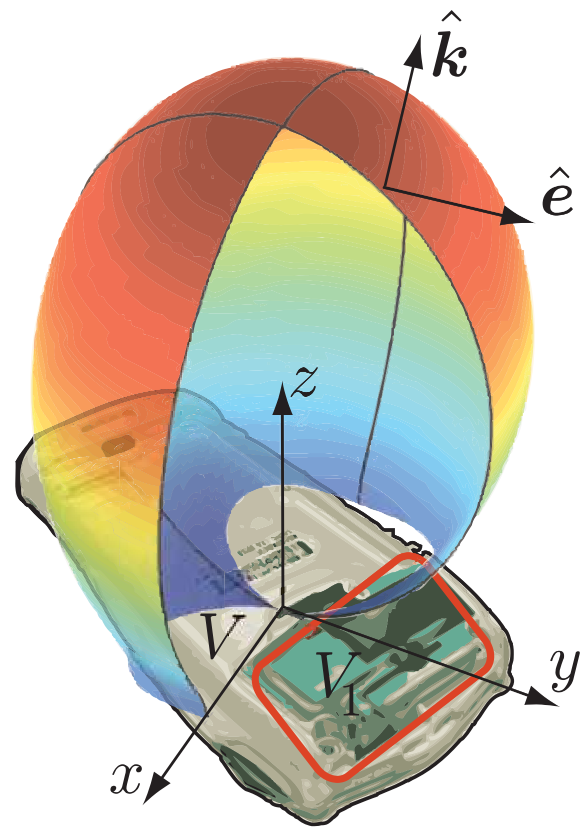
$$\begin{aligned} &\text{minimize} && \text{Stored energy} \\ &\text{subject to} && \text{Radiation intensity} = P_0 \end{aligned}$$

Q for superdirectivity $D \geq D_0$.

$$\begin{aligned} &\text{minimize} && \text{Stored energy} \\ &\text{subject to} && \text{Radiation intensity} = D_0 P_{\mathrm{rad}}/(4\pi) \\ &&& \text{Radiated power} \leq P_{\mathrm{rad}} \end{aligned}$$

Embedded structures

$$\begin{aligned} &\text{minimize} && \text{Stored energy} \\ &\text{subject to} && \text{Radiation intensity} = P_0 \\ &&& \text{Correct induced currents} \end{aligned}$$

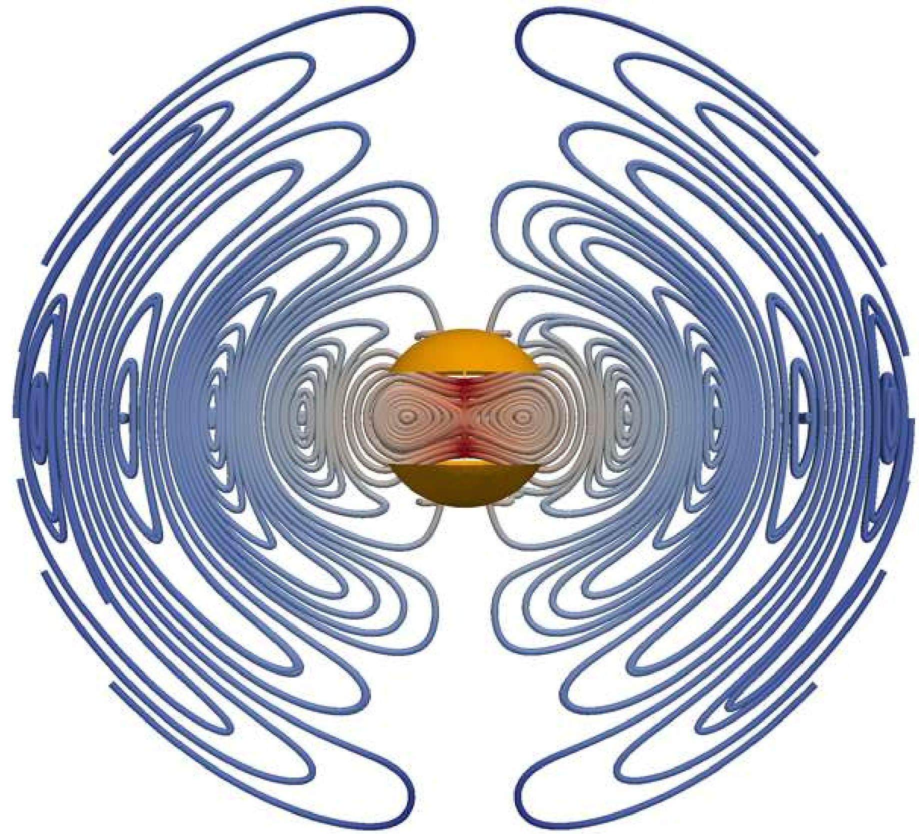


Need to:

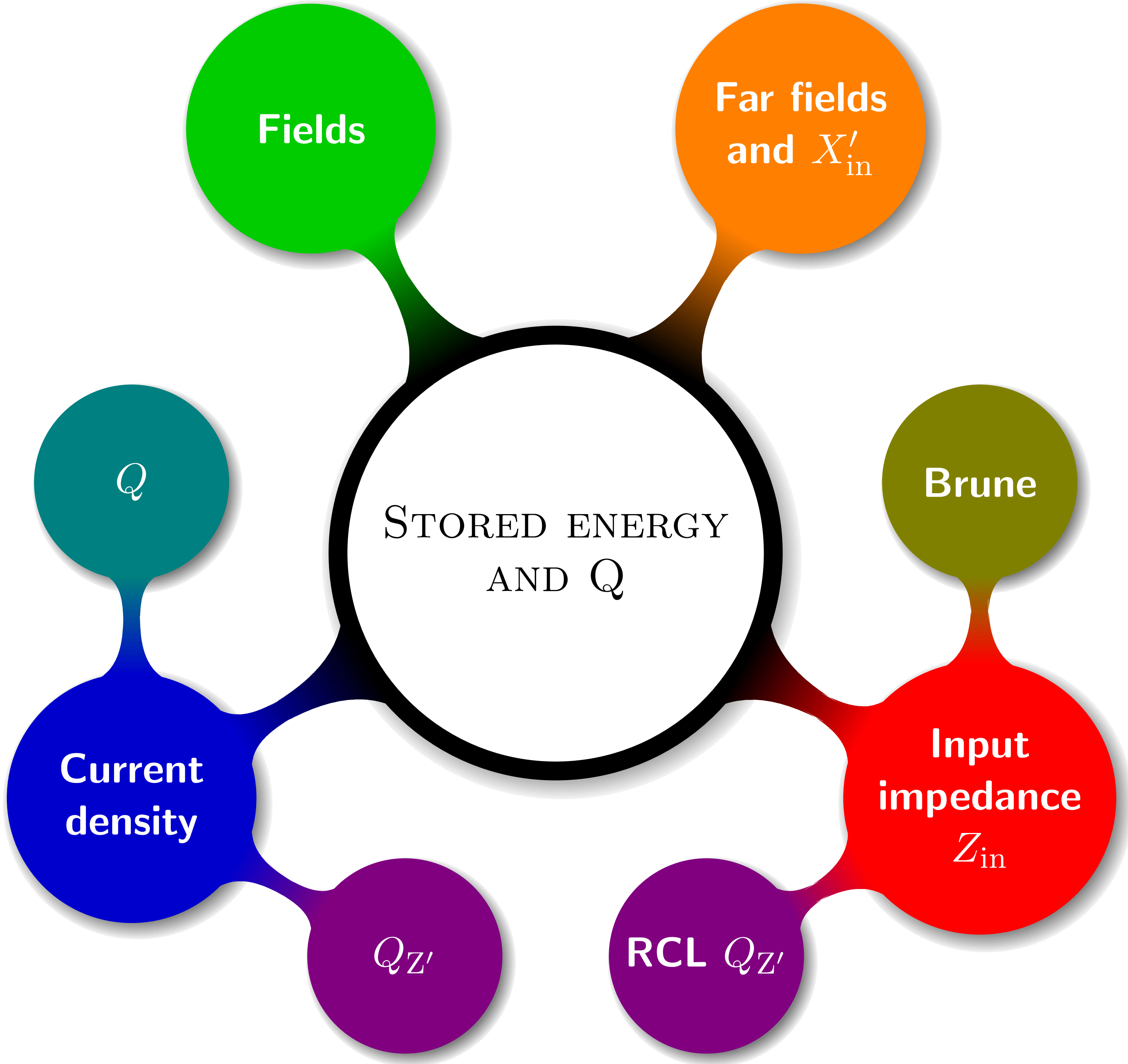
1. Express the *stored energy* in the current density $\boldsymbol{J}$.
2. Solve the optimization problems.

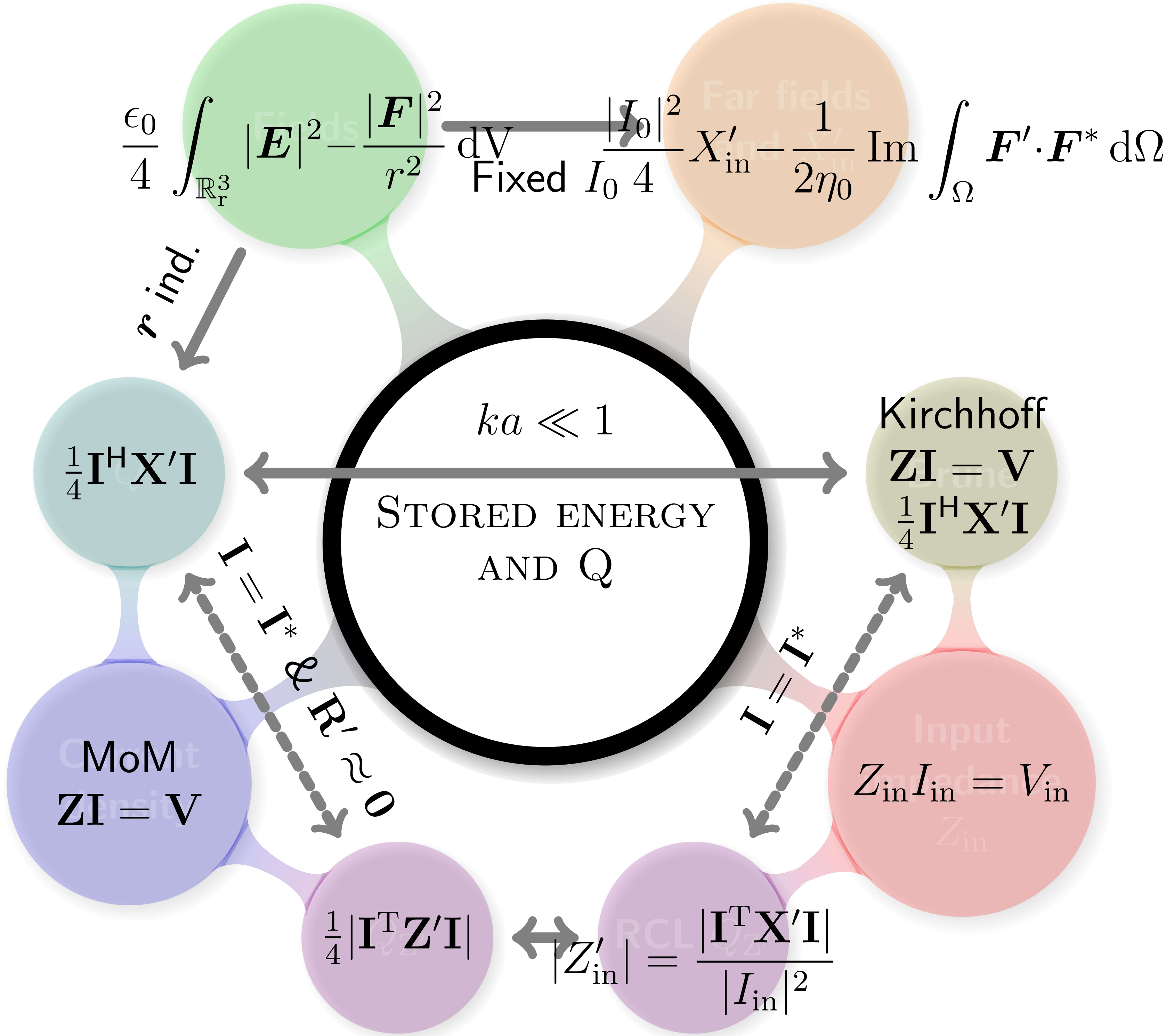
Stored electromagnetic energy

- Where is the energy stored?
  - Fields
  - Currents
  - Feed structure
- Stored according to what?
  - Input impedance
  - Material
  - Scatterer
- Why are we interested?
  - Basics physics
  - Antenna bandwidth
  - Physical bounds



There are several proposals for the stored energy in the literature. They agree for many cases but differ for some. Differences often due to different interpretations, assumptions, and applications.


Stored energy and Q
Fields
Far fields and X'_{in}
Brune
Input impedance Z_{in}
RCL $Q_{Z'}$
$Q_{Z'}$
Current density
Q



$\frac{\epsilon_0}{4}\int_{\mathbb{R}^3_r} |\boldsymbol{E}|^2 - \frac{|\boldsymbol{F}|^2}{r^2}\,\mathrm{dV}$
Fixed I_0
$\frac{|I_0|^2}{4} X'_{\mathrm{in}} - \frac{1}{2\eta_0}\,\mathrm{Im}\int_\Omega \boldsymbol{F}'\cdot\boldsymbol{F}^*\,\mathrm{d}\Omega$
$\boldsymbol{r}$ ind.
$\frac{1}{4}\mathbf{I}^{\mathrm{H}}\mathbf{X}'\mathbf{I}$
$ka \ll 1$
Stored energy and Q
Kirchhoff
$\mathbf{ZI} = \mathbf{V}$
$\frac{1}{4}\mathbf{I}^{\mathrm{H}}\mathbf{X}'\mathbf{I}$
$\mathbf{I} = \mathbf{I}^* \;\&\; \mathbf{R}' \approx \mathbf{0}$
$\mathbf{I} = \mathbf{I}^*$
MoM
$\mathbf{ZI} = \mathbf{V}$
$Z_{\mathrm{in}} I_{\mathrm{in}} = V_{\mathrm{in}}$
$\frac{1}{4}|\mathbf{I}^{\mathrm{T}}\mathbf{Z}'\mathbf{I}|$
$|Z'_{\mathrm{in}}| = \frac{|\mathbf{I}^{\mathrm{T}}\mathbf{X}'\mathbf{I}|}{|I_{\mathrm{in}}|^2}$

From MoM to stored energy (I)

Use a method of moments (MoM) formulation of the electric field integral equation (EFIE). Impedance matrix $\mathbf{Z} = \mathbf{R} + \mathrm{j}\mathbf{X}$

$$\frac{Z_{mn}}{\eta} = \mathrm{j} \int_{\Omega}\int_{\Omega} \left(k\boldsymbol{\psi}_{m1} \cdot \boldsymbol{\psi}_{n2} - \frac{1}{k}\nabla_1 \cdot \boldsymbol{\psi}_{m1} \nabla_2 \cdot \boldsymbol{\psi}_{n2} \right) \frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi r_{12}} \,\mathrm{dS}_1 \,\mathrm{dS}_2$$

where $\boldsymbol{\psi}_{n1} = \boldsymbol{\psi}_n(\boldsymbol{r}_1)$, $\boldsymbol{\psi}_{n2} = \boldsymbol{\psi}_n(\boldsymbol{r}_2)$, $m, n = 1, ..., N$, and $r_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$. The current density is $\boldsymbol{J}(\boldsymbol{r}) = \sum_{n=1}^{N} I_n \boldsymbol{\psi}_n(\boldsymbol{r})$ with the expansion coefficients determined from $\mathbf{ZI} = \mathbf{V}$, where $\mathbf{V}$ is a column matrix with the excitation coefficients.

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Differentiate the MoM impedance matrix

$$\frac{k\,\partial Z_{mn}}{\eta\,\partial k}=\int_{\Omega}\int_{\Omega}\mathrm{j}\Big(k\boldsymbol{\psi}_{m1}\cdot\boldsymbol{\psi}_{n2}+\frac{1}{k}\nabla_1\cdot\boldsymbol{\psi}_{m1}\nabla_2\cdot\boldsymbol{\psi}_{n2}\Big)\frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi r_{12}}$$
$$+k\Big(k\boldsymbol{\psi}_{m1}\cdot\boldsymbol{\psi}_{n2}-\frac{1}{k}\nabla_1\cdot\boldsymbol{\psi}_{m1}\nabla_2\cdot\boldsymbol{\psi}_{n2}\Big)\frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi}\,\mathrm{dS}_1\,\mathrm{dS}_2$$

From MoM to stored energy (I)

Use a method of moments (MoM) formulation of the electric field integral equation (EFIE). Impedance matrix $\mathbf{Z} = \mathbf{R} + \mathrm{j}\mathbf{X}$

$$\frac{Z_{mn}}{\eta} = \mathrm{j}\int_{\Omega}\int_{\Omega}\Big(\overset{W_{\mathrm{m}}}{k\boldsymbol{\psi}_{m1}\cdot\boldsymbol{\psi}_{n2}} - \frac{1}{k}\overset{W_{\mathrm{e}}}{\nabla_1\cdot\boldsymbol{\psi}_{m1}\nabla_2\cdot\boldsymbol{\psi}_{n2}}\Big)\frac{\mathrm{e}^{-\mathrm{j}kr_{12}}}{4\pi r_{12}}\,\mathrm{dS}_1\,\mathrm{dS}_2$$

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From MoM to stored energy (II)

Standard MoM implementations of the EFIE are easily modified to compute the stored energies. The sum and differences

$$W_{\mathrm{m}} + W_{\mathrm{e}} = \frac{1}{4}\mathbf{I}^{\mathsf{H}}\mathbf{X}'\mathbf{I} \quad \text{and } W_{\mathrm{m}} - W_{\mathrm{e}} = \frac{1}{4\omega}\mathbf{I}^{\mathsf{H}}\mathbf{X}\mathbf{I}$$

gives the stored magnetic and electric energies

$$W_{\mathrm{m}} = \frac{1}{8}\mathbf{I}^{\mathsf{H}}\left(\frac{\partial \mathbf{X}}{\partial \omega} + \frac{\mathbf{X}}{\omega}\right)\mathbf{I} \quad \text{and } W_{\mathrm{e}} = \frac{1}{8}\mathbf{I}^{\mathsf{H}}\left(\frac{\partial \mathbf{X}}{\partial \omega} - \frac{\mathbf{X}}{\omega}\right)\mathbf{I},$$

respectively. Electric $\mathbf{X}_{\mathrm{e}}$, and magnetic $\mathbf{X}_{\mathrm{m}}$, reactance matrices

$$\mathbf{X}_{\mathrm{e}} = \frac{1}{2}\left(\omega\mathbf{X}' - \mathbf{X}\right) \quad \text{and } \mathbf{X}_{\mathrm{m}} = \frac{1}{2}\left(\omega\mathbf{X}' + \mathbf{X}\right)$$

Identical to the stored energy expression (free space) introduced by Vandenbosch 2010 and already considered by Harrington and Mautz 1972.

Matrix expressions for the stored EM energies

Method of Moments approximation (expand $\boldsymbol{J}$ in basis functions)

$$W_{\mathrm{e}} \approx \frac{1}{4\omega}\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \quad \text{stored E-energy, } \mathbf{X}_{\mathrm{e}} \text{ electric reactance}$$

$$W_{\mathrm{m}} \approx \frac{1}{4\omega}\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \quad \text{stored M-energy, } \mathbf{X}_{\mathrm{m}} \text{ magnetic reactance}$$

$$P_{\mathrm{rad}} \approx \frac{1}{2}\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I} \quad \text{radiated power}$$

giving $\mathbf{Z} = \mathbf{R} + \mathrm{j}(\mathbf{X}_{\mathrm{m}} - \mathbf{X}_{\mathrm{e}})$. We also use

$$\hat{e}^{*} \cdot \boldsymbol{F} \approx \mathbf{F}\mathbf{I} \quad \text{far field}$$

$$\boldsymbol{E} \approx \mathbf{N}\mathbf{I} \quad \text{near field}$$

$$\mathbf{I}_{\mathrm{G}} \approx \mathbf{C}\mathbf{I}_{\mathrm{A}} \quad \text{induced current on a PEC}$$

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Pre-computed matrices used in the optimization.

Optimization of the current distribution

Characteristic modes

Modes with small Rayleigh quotients

$$\frac{\mathbf{I}^{\mathsf{H}}\mathbf{X}\mathbf{I}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}} = \frac{\mathbf{I}^{\mathsf{H}}(\mathbf{X}_{\mathrm{m}} - \mathbf{X}_{\mathrm{e}})\mathbf{I}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}}$$

Eigenvalue problem

$$(\mathbf{X}_{\mathrm{m}} - \mathbf{X}_{\mathrm{e}})\mathbf{I} = \nu\mathbf{R}\mathbf{I}$$

- Modes with low reactive power.
- Resonances ($\nu = 0$)
- Does not enforce low stored energy.

Stored energy

Minimize the energy Rayleigh quotient

$$\frac{\mathbf{I}^{\mathsf{H}}(\mathbf{X}_{\mathrm{m}} + \mathbf{X}_{\mathrm{e}})\mathbf{I}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}}$$

Eigenvalue problem

$$(\mathbf{X}_{\mathrm{m}} + \mathbf{X}_{\mathrm{e}})\mathbf{I} = \nu\mathbf{R}\mathbf{I}$$

- Modes with low stored energy.
- Does not enforce resonance.

Q-factor

Minimize the Q-factor quotient

$$\frac{2\max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}\}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}\mathbf{I}}$$

- Currents with low Q-factors.
- Resonance by tuning.
- Need to solve these optimization problems $\Rightarrow$ convex optimization.

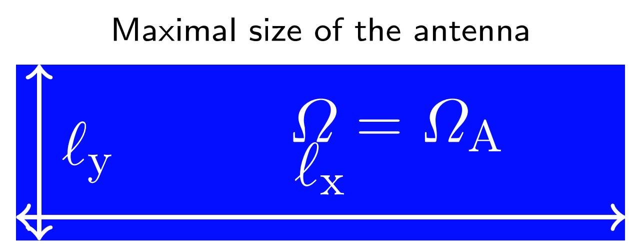
Chen and Wang 2015; Garbacz and Turpin 1971; Harrington and Mautz 1971

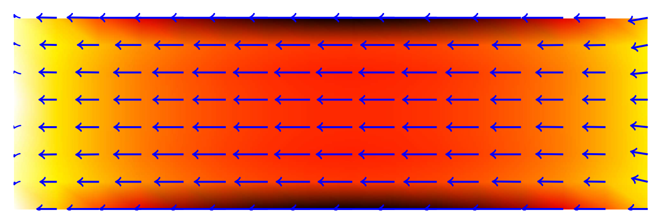
Currents for maximal G/Q

Determine a current density $\boldsymbol{J}(\boldsymbol{r})$ in the volume Ω that maximizes the partial-gain Q-factor quotient $G(\hat{\boldsymbol{k}},\hat{\mathrm{e}})/Q$.

- Partial radiation intensity $P(\hat{\boldsymbol{k}},\hat{\mathrm{e}})$

$$\frac{G(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{Q} = \frac{2\pi P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{c_0 k \max\{W_{\mathrm{e}}, W_{\mathrm{m}}\}}.$$



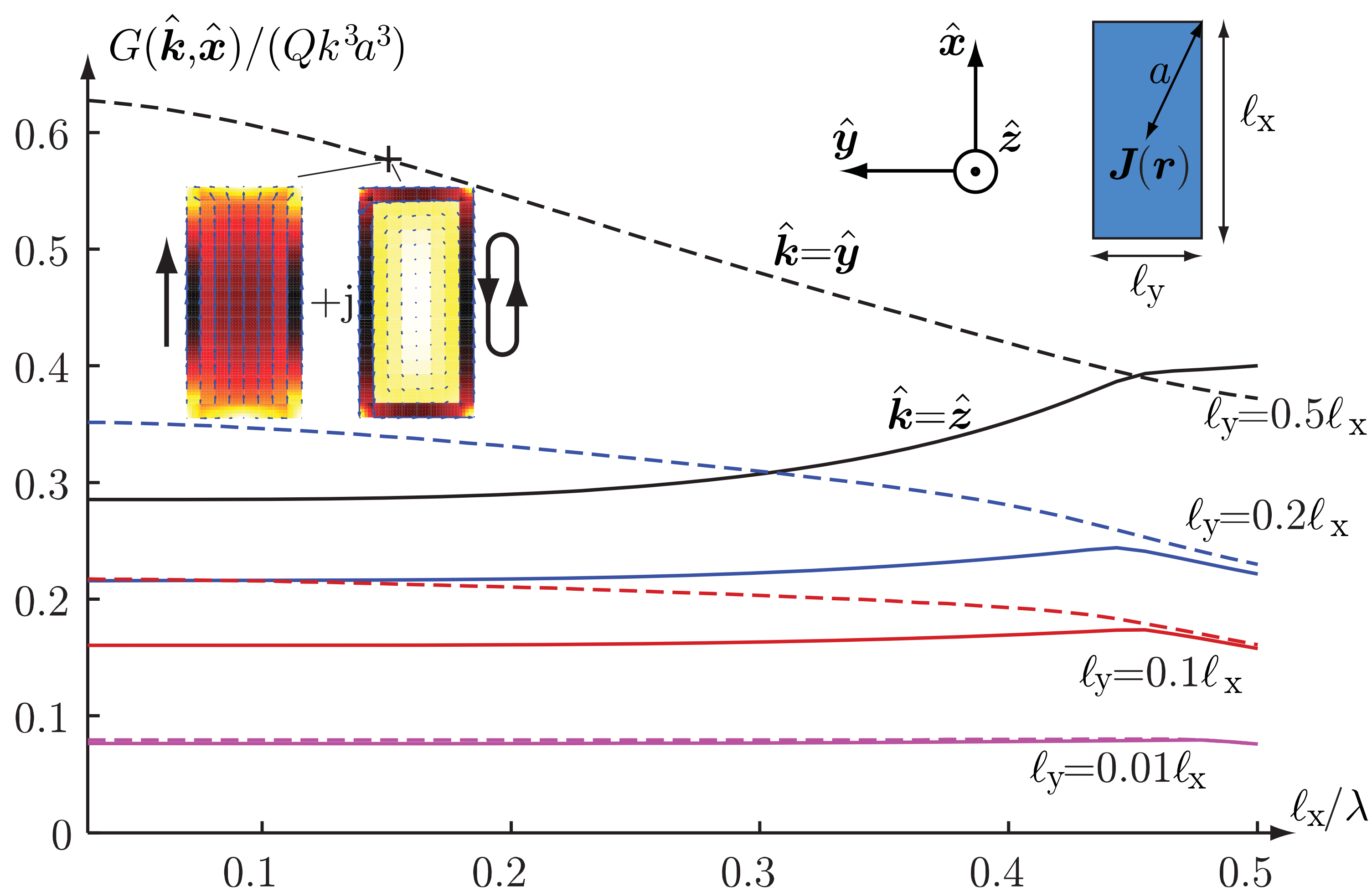


- Scale $\boldsymbol{J}$ and reformulate $\max. P$ as $\max. \operatorname{Re}\{\hat{\boldsymbol{e}}^* \cdot \boldsymbol{F}\}$.
- Convex optimization problem.

$$\begin{aligned} &\text{maximize} && \operatorname{Re}\{\mathbf{FI}\} \\ &\text{subject to} && \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \leq 1 \\ & && \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \leq 1 \end{aligned}$$

Determines a current density $\boldsymbol{J}(\boldsymbol{r})$ in the region Ω with maximal partial radiation intensity and unit stored EM energy.

Maximal $G(\hat{\boldsymbol{k}}, \hat{\boldsymbol{x}})/Q$ for planar rectangles



Solution for current densities confined to planar rectangles with side lengths ℓ_x and $\ell_y = \{0.01, 0.1, 0.2, 0.5\}\ell_x$.

M. Gustafsson, *etal*, Tutorial on antenna current optimization using MATLAB and CVX, TEAT-7241, 1-62, 2015

G/Q bounds

Typical (but not optimal) MATLAB code using CVX

```
cvx_begin
   variable I(n) complex;        % current density
   maximize(real(F*I))           % far-field
   subject to
      quad_form(I,Xe) <= 1;      % stored E energy
      quad_form(I,Xm) <= 1;      % stored M energy
cvx_end
```

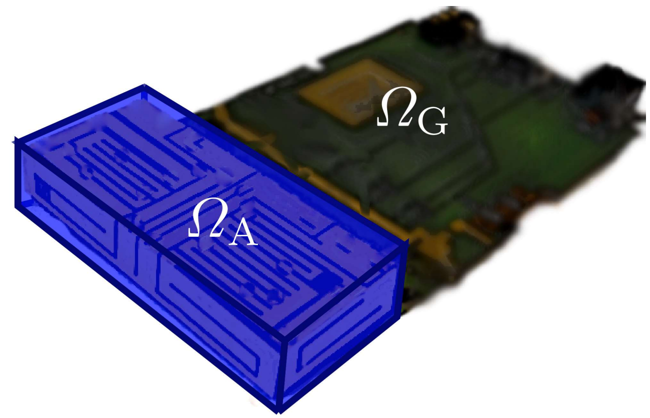
- Similar to the forward scattering bounds (2007) for TM.
- Can design 'optimal' electric dipole mode (TM) antennas.
- TE modes and TE+TM are not well understood.

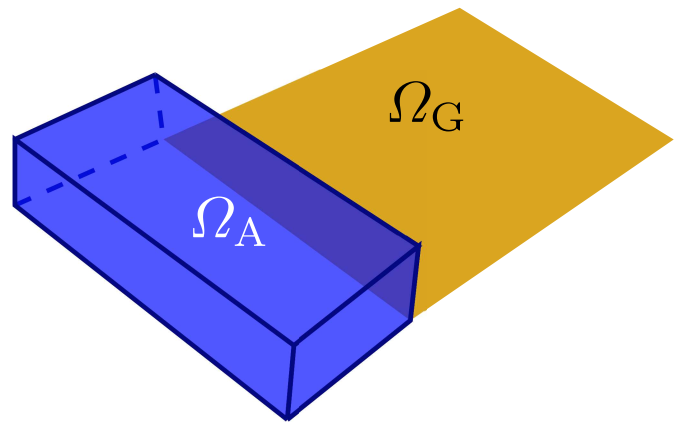
We can reformulate the complex optimization problem to analyze superdirectivity, antennas with a prescribed radiation pattern, losses, and antennas embedded in a PEC structure.

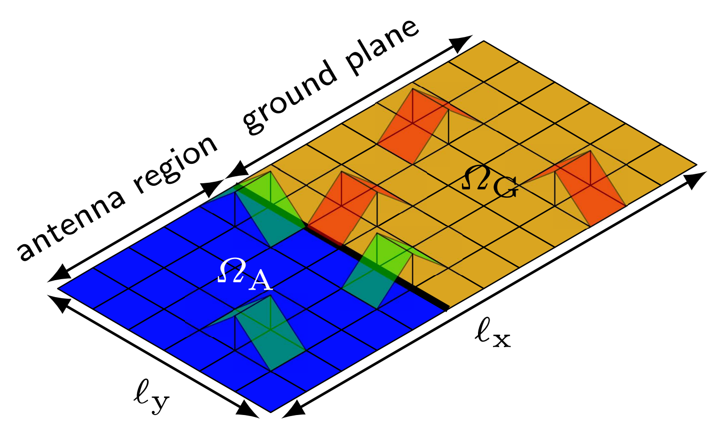
M. Gustafsson, *etal*, Tutorial on antenna current optimization using MATLAB and CVX, TEAT-7241, 1-62, 2015

Optimal performance for embedded antennas

- Common with antennas embedded in metallic structures.
- The induced currents radiate but they are not arbitrary.
- Linear map from the antenna region adds a (convex) constraint.
- Here, we assume that the surrounding structure is PEC and add a constraint to account for the induced currents on the surrounding structure in the G/Q formulation.







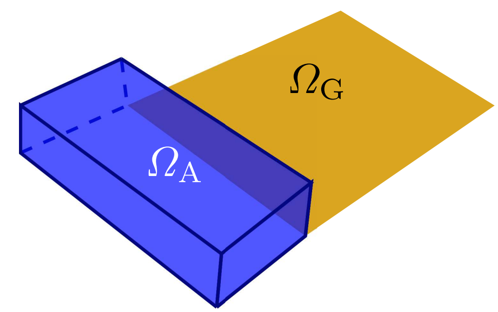
Currents for maximal G/Q for embedded antennas

Determine an optimal current density $\boldsymbol{J}_{\mathrm{A}}(\boldsymbol{r})$ in the region Ω_{A}. Assume that the ground plane $\Omega_{\mathrm{G}} = \Omega \setminus \Omega_{\mathrm{A}}$ is PEC.

Can minimize the stored energy for given radiated field

$$\begin{aligned} \text{minimize} \quad & \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ \text{subject to} \quad & \mathbf{F}\mathbf{I} = 1 \\ & \mathbf{I}_{\mathrm{G}} = \mathbf{C}\mathbf{I}_{\mathrm{A}} \end{aligned}$$

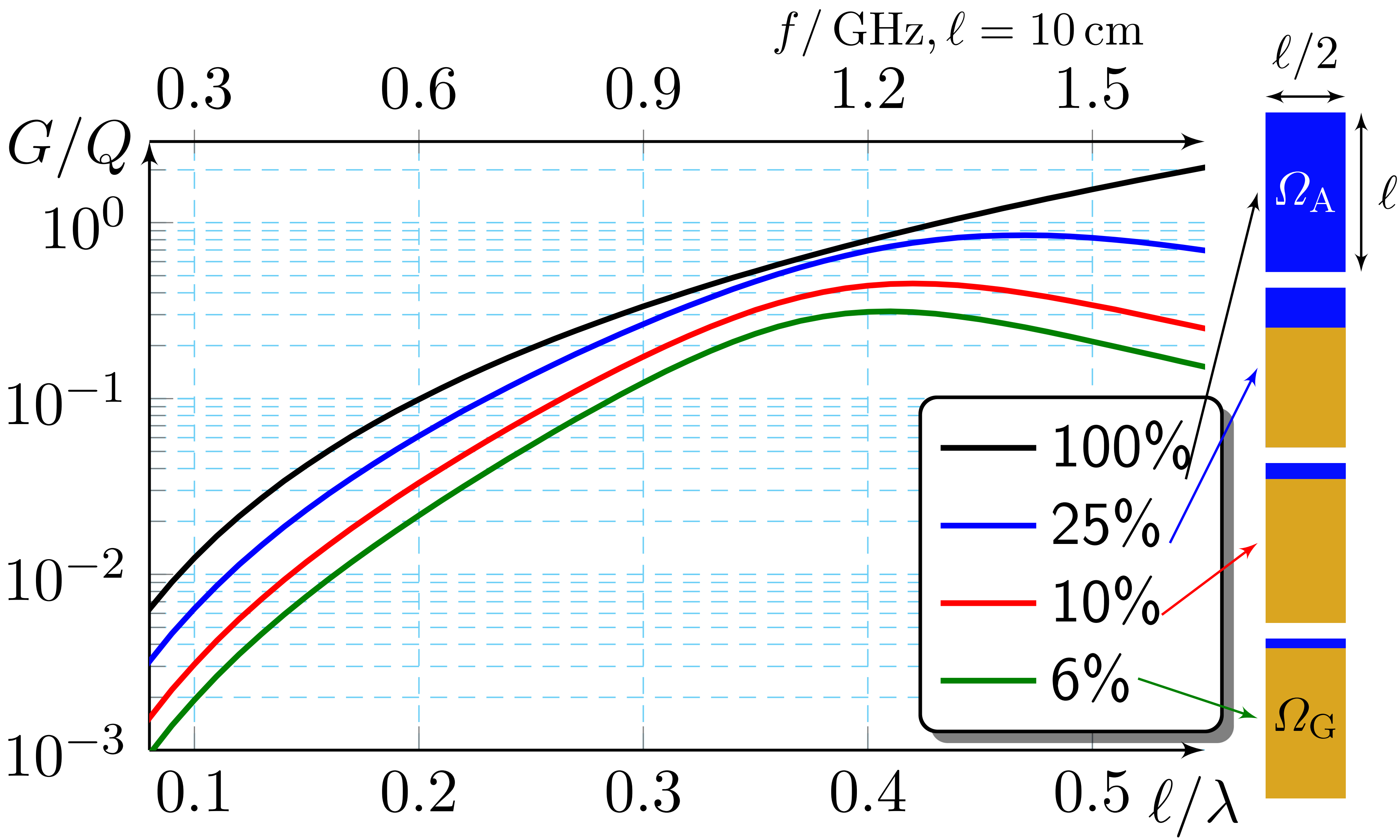
or maximize the radiated field for given stored energy



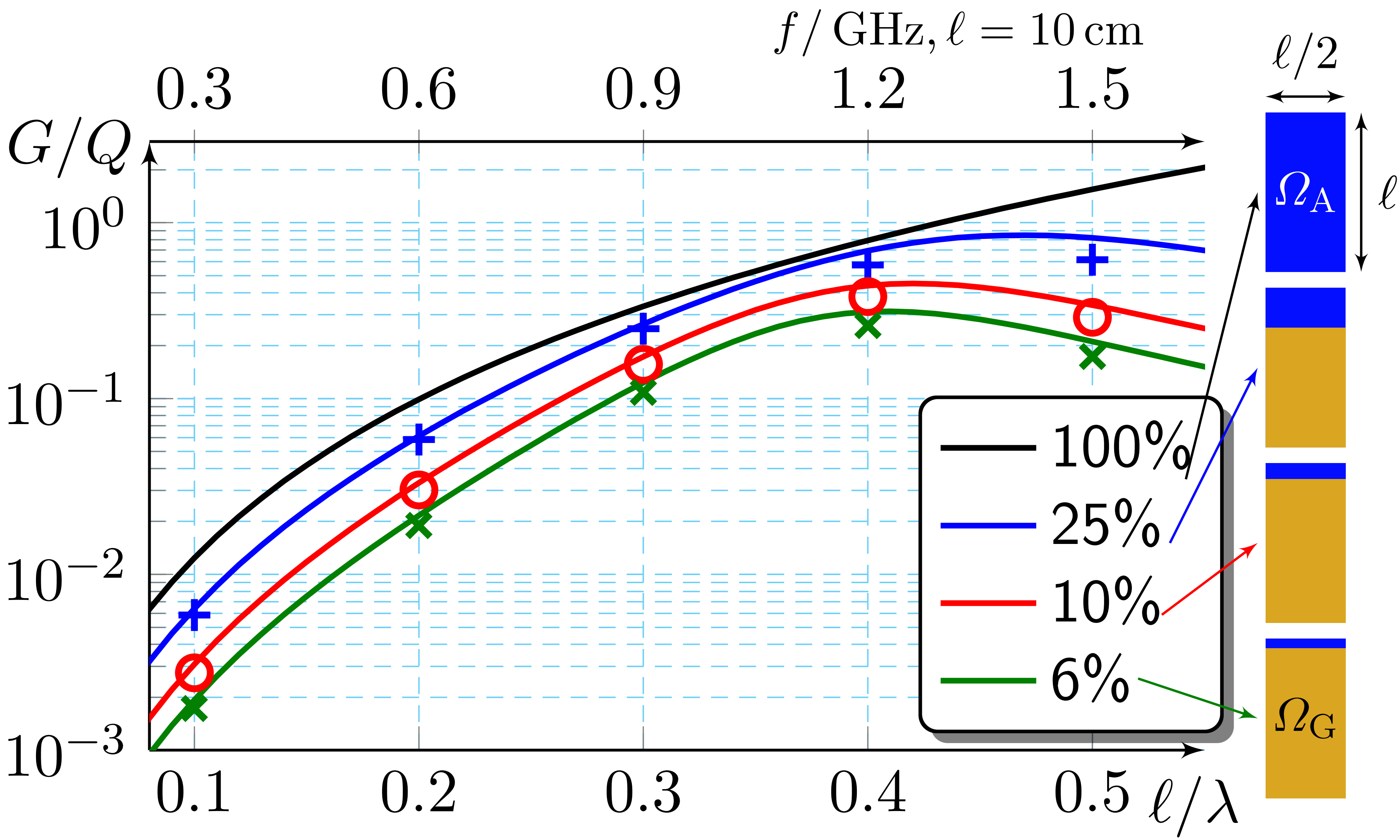
$$\begin{aligned} \text{maximize} \quad & \mathrm{Re}\{\mathbf{F}\mathbf{I}\} \\ \text{subject to} \quad & \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \leq 1 \\ & \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \leq 1 \\ & \mathbf{I}_{\mathrm{G}} = \mathbf{C}\mathbf{I}_{\mathrm{A}} \end{aligned}$$

Can also eliminate $\mathbf{I}_{\mathrm{G}}$.

Finite ground plane with $\{6, 10, 25, 100\}\%$ antenna region



Finite ground plane with $\{6, 10, 25, 100\}\%$ antenna region

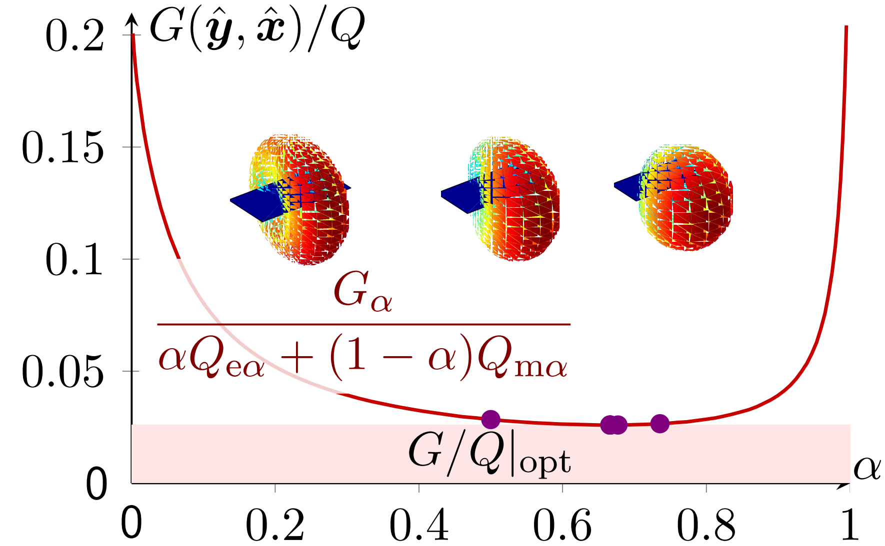


Why convex optimization: illustration

The upper bound on $G/Q|_{\text{opt}}$ is obtained by solving the dual (relaxed) problem, *i.e.*, finding the minimum of the (red) curve

$$\frac{G}{Q}\bigg|_{\text{opt}} \leq \frac{G_\alpha}{\alpha Q_{\text{e}\alpha} + (1-\alpha)Q_{\text{m}\alpha}}$$

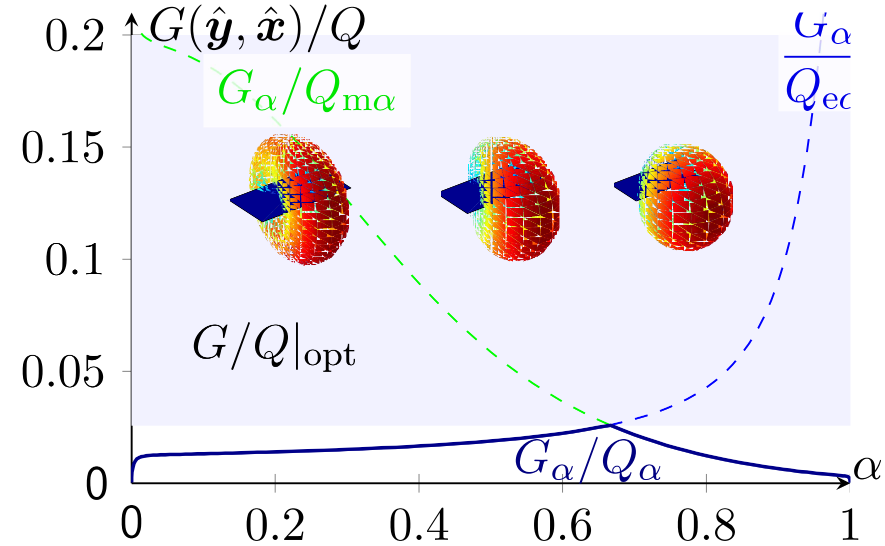
Efficiently solved with Newton iterations (cost $\mathbf{Ax} = \mathbf{b}$ per it).



$\ell/\lambda \approx 0.1$ or $ka \approx 0.35$

The Newton iterations converge as $\alpha \approx 0.5$, 0.73536, 0.67677, 0.66629, 0.66602, 0.66602.

Why convex optimization: illustration



$\ell/\lambda \approx 0.1$ or $ka \approx 0.35$

For free we also compute G/Q for the (dual) current $\mathbf{I}_\alpha$ to get

$$\frac{G_\alpha}{\max\{Q_{\mathrm{e}\alpha}, Q_{\mathrm{m}\alpha}\}} \leq \left.\frac{G}{Q}\right|_{\mathrm{opt}}$$

Why convex optimization: illustration

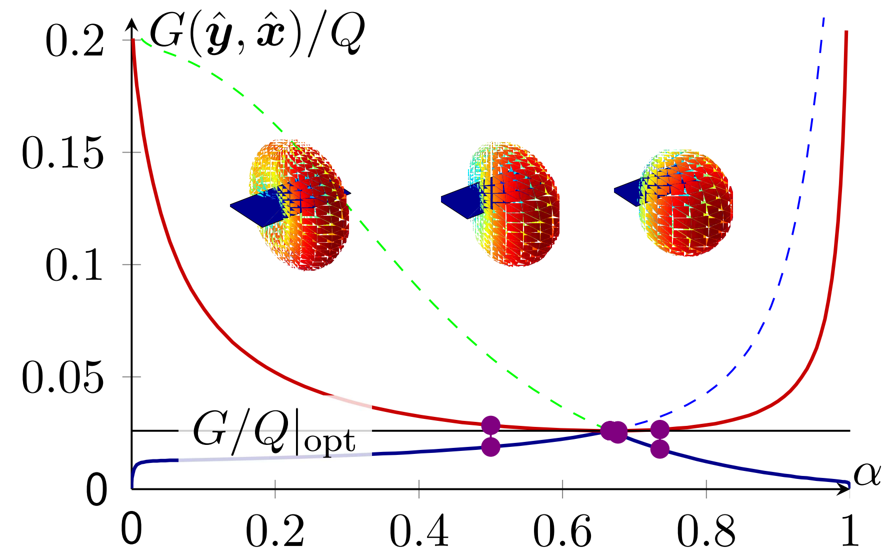
The upper bound on $G/Q|_{\mathrm{opt}}$ is obtained by solving the dual (relaxed) problem, *i.e.*, finding the minimum of the (red) curve

$$\frac{G}{Q}\bigg|_{\mathrm{opt}} \leq \frac{G_\alpha}{\alpha Q_{\mathrm{e}\alpha} + (1-\alpha)Q_{\mathrm{m}\alpha}}$$

Efficiently solved with Newton iterations (cost $\mathbf{A}\mathbf{x} = \mathbf{b}$ per it).

For free we also compute G/Q for the (dual) current $\mathbf{I}_\alpha$ to get

$$\frac{G_\alpha}{\max\{Q_{\mathrm{e}\alpha}, Q_{\mathrm{m}\alpha}\}} \leq \frac{G}{Q}\bigg|_{\mathrm{opt}}$$



$\ell/\lambda \approx 0.1$ or $ka \approx 0.35$

The Newton iterations converge as $\alpha \approx 0.5$, 0.73536, 0.67677, 0.66629, 0.66602, 0.66602. Duality gap in G/Q approximately $10^{-\{2,2,3,4,8,16\}}$.

Why convex optimization: illustration

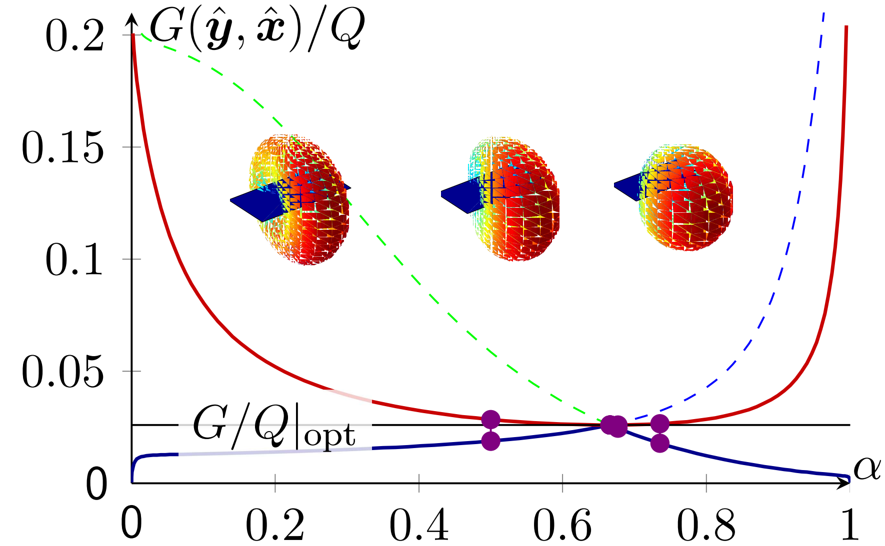
The upper bound on $G/Q|_{\mathrm{opt}}$ is obtained by solving the dual (relaxed) problem, *i.e.*, finding the minimum of the (red) curve

$$\frac{G}{Q}\bigg|_{\mathrm{opt}} \leq \frac{G_\alpha}{\alpha Q_{\mathrm{e}\alpha} + (1-\alpha)Q_{\mathrm{m}\alpha}}$$

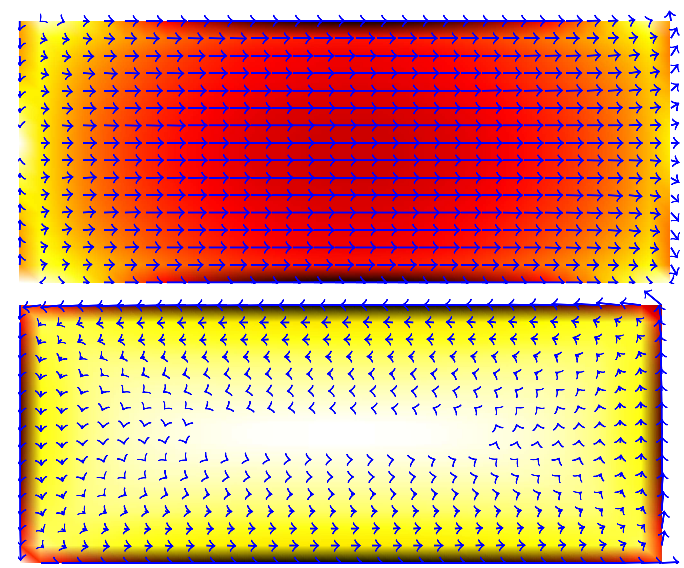
Efficiently solved with Newton iterations (cost $\mathbf{Ax}=\mathbf{b}$ per it).

For free we also compute G/Q for the (dual) current $\mathbf{I}_\alpha$ to get

$$\frac{G_\alpha}{\max\{Q_{\mathrm{e}\alpha}, Q_{\mathrm{m}\alpha}\}} \leq \frac{G}{Q}\bigg|_{\mathrm{opt}}$$



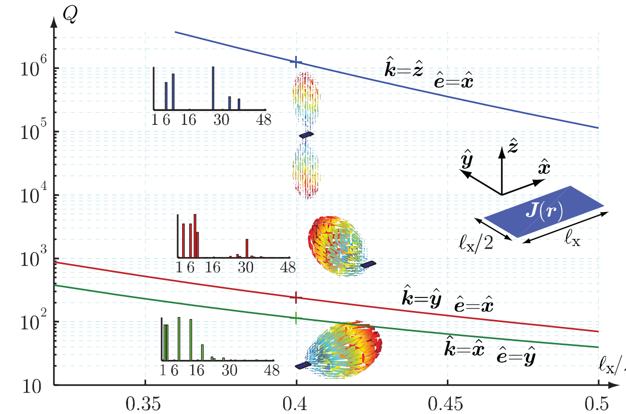
$\ell/\lambda \approx 0.1$ or $ka \approx 0.35$



Why: simple optimization formulations

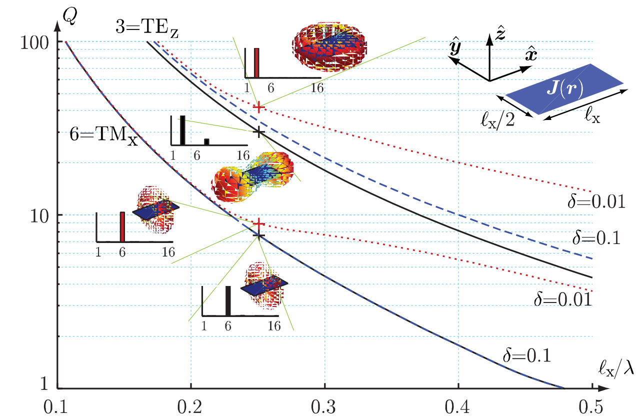
Super directivity:

$$\begin{aligned}\text{minimize} \quad & \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ \text{subject to} \quad & \mathbf{F}\mathbf{I} = 1 \\ & \mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I} \leq 4\pi/(\eta_0 D_0)\end{aligned}$$



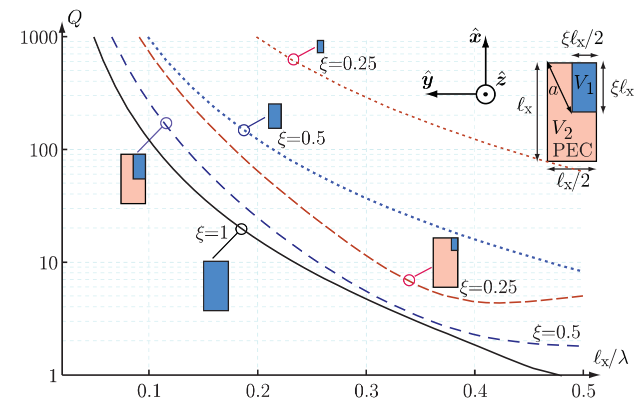
Prescribed far field:

$$\begin{aligned}\text{minimize} \quad & \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ \text{subject to} \quad & \int_{\Omega} |\boldsymbol{F}(\hat{\boldsymbol{k}}) - \boldsymbol{F}_0(\hat{\boldsymbol{k}})|^2 \,\mathrm{d}\Omega_{\hat{\boldsymbol{k}}} < \delta\end{aligned}$$



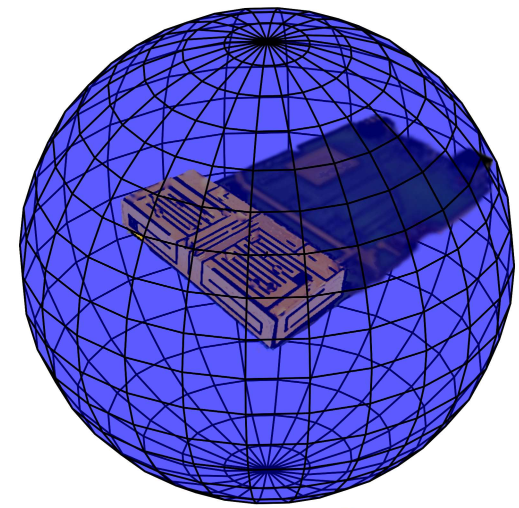
Embedded antennas:

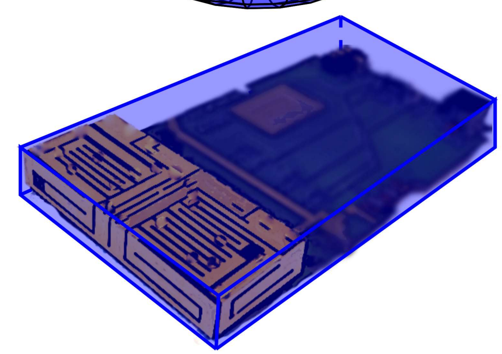
$$\begin{aligned}\text{minimize} \quad & \max\{\mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}, \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}\} \\ \text{subject to} \quad & \mathbf{F}\mathbf{I} = 1 \\ & \mathbf{I}_{\mathrm{G}} = \mathbf{C}\mathbf{I}_{\mathrm{A}}\end{aligned}$$

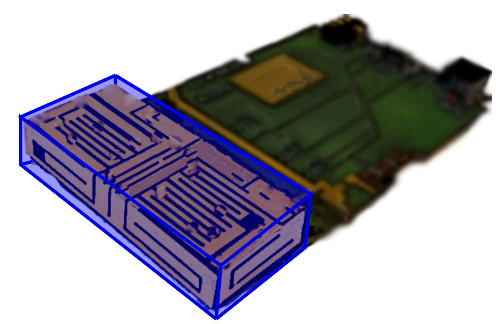


Summary

- Physical bounds from spheres (Chu 1948) and arbitrary shapes (Gustafsson *etal* 2007) to embedded antennas...
- Stored energy in the current density.
- Optimization of the antenna structure (global optimization) and the antenna currents (convex optimization).
- Convex optimization for bounds and optimal currents: G/Q, superdirective, embedded, ...
- Closed form solutions for small antennas.
- Non-Foster to overcome $B \sim 1/Q$ ...







Initial results for efficiency, more realistic geometries (phones), SAR, MIMO. Investigating dielectrics, volume currents, magnetic currents, ...

Slides: `http://www.eit.lth.se/staff/mats.gustafsson`

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